Question: How does the language of mathematics help us model the motion of objects through time and space?

Chapter 4: 1-D Kinematics (Equations of Motion)

We can study motion even if we do not have a graph. In fact, we often don't have a graph. In that case, we are going to us equations to study motion.

→ straight line, constant accel.

In this chapter, we will study uniformly accelerated rectilinear motion. It is motion in a straight line, with 2 possible directions (ex: forward/backward, up/down) with constant acceleration.

The equations we will use in this chapter are only valid for **uniformly accelerated motion** (i.e. constant acceleration).

The five quantities

When studying motion over a certain period of time, we work with 5 physical quantities (5 variables). They are:

Δt: time (in s)

$\Delta \vec{d}$: displacement (m)

$\vec{v}_i$: initial velocity (m/s)

$\vec{v}_f$: final velocity (m/s)

$\vec{a}$: acceleration (m/s²)

→ drop arrow on top

Note: We will not be using vector notation here, but we are still going to use "+" and "–" to distinguish between the 2 different possible directions.

The five equations:

We will derive the equations relating the above quantities (4 at a time).

First equation: The definition of average acceleration.

$$\vec{a}_{ave} = \frac{\vec{v}_f - \vec{v}_i}{\Delta t}$$

because constant accel, $a = \vec{a}_{ave}$

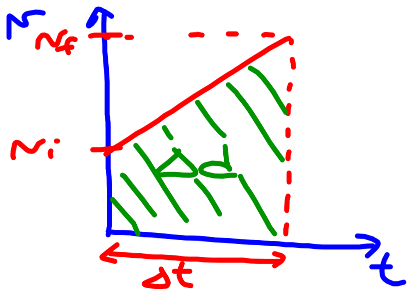
$$a = \frac{v_f - v_i}{\Delta t}$$

$$a(\Delta t) = v_f - v_i$$

$$a(\Delta t) + v_i = v_f$$

① $$\boxed{v_f = v_i + a\Delta t}$$ *no Δd

Second equation: Finding the displacement from a v-t graph.



$$\Delta d = \text{area} = \frac{(b+B)h}{2}$$

$$\Delta d = \frac{(v_i + v_f)\Delta t}{2}$$

$$\boxed{\Delta d = \frac{(v_i + v_f)}{2}\Delta t}$$ no a!

Examples:

1. A car is going at a constant speed of 12 m/s. It then accelerates for 5.0 s at a rate of 3.0 m/s^2. What is the final velocity of the car?

$v_i = 12\ m/s$
$\Delta t = 5.0\ s$
$a = 3.0\ m/s^2$
$v_f = ?$

$$v_f = v_i + a\Delta t$$
$$= 12\ m/s + (3.0\ \frac{m}{s^2})(5.0\ s)$$
$$= \underbrace{12\ \frac{m}{s}}_{0\ \text{dec}} + \underbrace{15\ \frac{m}{s}}_{0\ \text{dec}}$$
$$= 27\ m/s$$

units
$\frac{m}{\cancel{s^2}} \cdot \cancel{s}$

2. A cyclist decelerated from 15.0 m/s to 5.0 m/s at a rate of 2.0 m/s^2. How long did this take?

$v_i = 15.0\ m/s$
$v_f = 5.0\ m/s$
$a = -2.0\ m/s^2$
$\Delta t = ?$

$$v_f = v_i + a\Delta t$$
$$v_f - v_i = a\Delta t$$
$$\Delta t = \frac{v_f - v_i}{a}$$
$$= \frac{5.0\ m/s - 15.0\ m/s}{-2.0\ m/s^2}$$
$$= \frac{-10.0\ m/s}{-2.0\ m/s^2} = 5.0\ s$$

units
$\frac{\cancel{m}}{\cancel{s}} \times \frac{s^{\cancel{2}}}{\cancel{m}}$

3. A bus accelerates form rest over a distance of 125 m. What is the velocity of the bus after the acceleration if this took 10.0 s?

$v_i = 0$
$\Delta d = 125\ m$
$\Delta t = 10.0\ s$
$v_f = ?$

$$\Delta d = \frac{(\cancel{v_i}^{\,0} + v_f)}{2} \cdot \Delta t$$
$$\Delta d = \frac{v_f}{2} \cdot \Delta t$$
$$v_f = \frac{2 \cdot \Delta d}{\Delta t}$$
$$= \frac{2(125\ m)}{10.0\ s}$$
$$= 25.0\ m/s$$

Third Equation: Finding an equation that doesn't have v_f.

In the equation $\Delta d = \frac{(v_i + v_f)\Delta t}{2}$, replace v_f by $v_f = v_i + a\Delta t$.

$$\Delta d = \frac{(v_i + v_i + a\Delta t)\Delta t}{2}$$

$$\Delta d = \frac{(2v_i + a\Delta t)\Delta t}{2}$$

$$\Delta d = \frac{2v_i\Delta t + a(\Delta t)^2}{2}$$

$$\boxed{\Delta d = v_i\Delta t + \frac{1}{2}a(\Delta t)^2}$$ no v_f

Fourth Equation: Finding an equation that doesn't have v_i.

In the equation $\Delta d = \frac{(v_i + v_f)\Delta t}{2}$, replace v_i by $v_i = v_f - a\Delta t$.

$v_f = v_i + a\Delta t$

$v_i = v_f - a\Delta t$

$$\Delta d = \frac{(v_f - a\Delta t + v_f)\Delta t}{2}$$

$$\Delta d = \frac{(2v_f - a\Delta t)\Delta t}{2}$$

$$\Delta d = \frac{2v_f\Delta t - a(\Delta t)^2}{2}$$

$$\boxed{\Delta d = v_f\Delta t - \frac{1}{2}a(\Delta t)^2}$$ no v_i

Fifth Equation: Finding an equation that doesn't have Δt.

$$\Delta d = \frac{(v_i + v_f)\Delta t}{2}$$

$$v_f = v_i + a\Delta t$$

$$v_f - v_i = a\Delta t$$

$$\Delta t = \frac{v_f - v_i}{a}$$

$$\Delta d = \frac{(v_i + v_f)}{2} \cdot \frac{(v_f - v_i)}{a}$$

$$\Delta d = \frac{(v_f + v_i)(v_f - v_i)}{2a} \rightarrow \text{diff of squares}$$

$$\Delta d = \frac{v_f^2 - v_i^2}{2a}$$

$$2a\Delta d = v_f^2 - v_i^2$$

$$\boxed{v_f^2 = v_i^2 + 2a\Delta d} \quad \text{no } \Delta t$$

Examples:

1. A cyclist is traveling at a constant speed of 4.0 m/s. He then accelerates at a rate of 0.40 m/s² for 5.0 s. What distance does he cover while he accelerates?

$v_i = 4.0\,m/s$

$a = 0.40\,m/s^2$

$\Delta t = 5.0\,s$

$\Delta d = ?$

$$\Delta d = v_i\Delta t + \frac{1}{2}a(\Delta t)^2 \qquad \frac{m}{\cancel{s^2}} \times \cancel{s^2}$$

$$= (4.0\tfrac{m}{\cancel{s}})(5.0\cancel{s}) + \frac{1}{2}(0.40\tfrac{m}{s^2})(5.0s)^2$$

$$= 20.\,m + 5.0\,m$$

$$= 25\,m$$

2. A driver going 25 m/s sees a stalled car 125 m in front of him. At what rate must the car decelerate if he wants to avoid a crash?

$v_i = 25\,m/s$

$\Delta d = 125\,m$

$a = ?$

$\cancel{v_f = 0}$

~~Stop!~~

$$\cancel{v_f}^{2}\,^{0} = v_i^2 + 2a\Delta d$$

$$0 = v_i^2 + 2a\Delta d$$

$$-v_i^2 = 2a\Delta d$$

$$a = \frac{-v_i^2}{2\Delta d}$$

$$= \frac{-(25\,m/s)^2}{2(125\,m)}$$

$$a = -2.5\,m/s^2$$

↳ slows down at $2.5\,m/s^2$

units

$$\frac{\cancel{m^2}}{s^2} \times \frac{1}{\cancel{m}}$$

3. A bus starts from rest and accelerates at a rate of 1.5 m/s^2. How long does it take this bus to cover 55 m?

$v_i = 0$

$a = 1.5\,m/s^2$

$\Delta d = 55\,m$

$\Delta t = ?$

$$\Delta d = \cancel{v_i \Delta t}^{\,0} + \frac{1}{2}a(\Delta t)^2$$

$$\Delta d = \frac{1}{2}a(\Delta t)^2$$

$$\sqrt{(\Delta t)^2} = \sqrt{\frac{2\Delta d}{a}} \quad \text{OR} \quad = \frac{\Delta d}{\frac{1}{2}a}$$

$$\Delta t = \sqrt{\frac{2(55\,m)}{1.5\,m/s^2}}$$

$$\Delta t = 8.6\,s$$

units

$$\sqrt{\cancel{m} \times \frac{s^2}{\cancel{m}}} = s$$

4. How long does it take a runner to cover 23.5 m if he begins with a speed of 2.2 m/s and accelerates at a rate of 1.0 m/s^2?

$\Delta d = 23.5\,m$
$v_i = 2.2\,m/s$
$a = 1.0\,m/s^2$
$\Delta t = ?$

$\Delta d = v_i \Delta t + \frac{1}{2}a(\Delta t)^2$

solve the quadratic!
(only for eq. 3 or 4, with nothing = 0, and solve for Δt!)

Method #1 - Quadratic formula

$\Delta d = v_i \Delta t + \frac{1}{2}a(\Delta t)^2$

$23.5\,m = (2.2\frac{m}{s})(\Delta t) + \frac{1}{2}(1.0\frac{m}{s^2})(\Delta t)^2$

$0 = (2.2)(\Delta t) + (0.5)(\Delta t)^2 - 23.5$

$0 = \underbrace{(0.5)}_{a}(\Delta t)^2 + \underbrace{(2.2)}_{b}(\Delta t) \underbrace{- 23.5}_{c}$

$\Delta t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Delta t = \frac{-(2.2) \pm \sqrt{(2.2)^2 - 4(0.5)(-23.5)}}{2(0.5)}$

$\Delta t = \frac{-2.2 \pm \sqrt{51.84}}{1.0}$

$\Delta t = \frac{-2.2 \pm 7.2}{1.0}$

$+$: $\Delta t = \frac{-2.2 + 7.2}{1.0}$

$\boxed{\Delta t = 5.0\,s}$

accept

$-$: $\Delta t = \frac{-2.2 - 7.2}{1.0}$

$\Delta t = \cancel{-9.4}\,s$

reject: no neg. time!

Method #2 Find v_f first
(then use another equation)

$\Delta t = ?$ wait
$v_i = 2.2\,m/s$
$\Delta d = 23.5\,m$
$a = 1.0\,m/s^2$
$v_f = ?$

① Find v_f

$v_f^2 = v_i^2 + 2a\Delta d$

$= (2.2\frac{m}{s})^2 + 2(1.0\frac{m}{s^2})(23.5\,m)$

$v_f^2 = 4.84\frac{m^2}{s^2} + 47\frac{m^2}{s^2}$

$\sqrt{v_f^2} = \sqrt{51.84\frac{m^2}{s}}$

$v_f = \pm 7.2\,m/s$

$v_f = +7.2\,m/s$
(no change in direction)

② Find Δt

$v_f = 7.2\,m/s$
$v_i = 2.2\,m/s$
$\Delta t = ?$
$a = 1.0\,m/s^2$

$v_f = v_i + a\Delta t$

$v_f - v_i = a\Delta t$

$\Delta t = \frac{v_f - v_i}{a}$

$= \frac{7.2\frac{m}{s} - 2.2\frac{m}{s}}{1.0\,m/s^2}$

$\boxed{\Delta t = 5.0\,s}$

Summary of Equations

$$v_f = v_i + a\Delta t$$
$$\Delta d = \frac{(v_i + v_f)\,\Delta t}{2}$$
$$\Delta d = v_i\Delta t + \tfrac{1}{2}a(\Delta t)^2$$
$$\Delta d = v_f\Delta t - \tfrac{1}{2}a(\Delta t)^2$$
$$v_f^2 = v_i^2 + 2a\Delta d$$

Special Case 1: Constant Velocity

When the velocity is constant, $a = 0$.

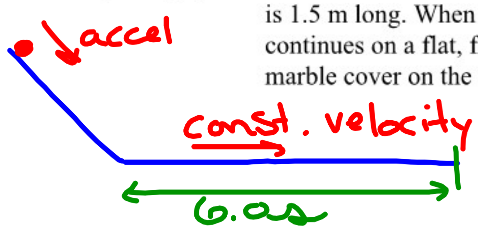
Equations 1 and 5 simplify to $v_f = v_i$.

Equations 2, 3, & 4 simplify to $\Delta d = v\Delta t$ (since $v = v_f = v_i$).

So, when dealing with constant velocity, we only need one equation:

$$\boxed{\Delta d = v\Delta t}$$ *only for constant velocity

Example: A marble accelerates from rest at a rate of 0.25 m/s^2 down an incline that is 1.5 m long. When it reaches the bottom of the incline, the marble continues on a flat, frictionless surface for 6.0 s. What distance does the marble cover on the flat surface?



① Accel (incline)

$\Delta d = 1.5\,m$

$a = 0.25\,m/s^2$

$v_i = 0$

$v_f = ?$

$v_f^2 = \cancel{v_i^2}^{\,0} + 2a\Delta d$

$\sqrt{v_f^2} = \sqrt{2(0.25\,\frac{m}{s^2})(1.5\,m)}$

$v_f = 0.866\,m/s$

② Const. vel (flat)

$v = 0.866\,m/s$

$\Delta t = 6.0\,s$

$\Delta d = ?$

$\Delta d = v\Delta t$

$= (0.866\,\frac{m}{\cancel{s}})(6.0\,\cancel{s})$

$\boxed{\Delta d = 5.2\,m}$

Special Case 2: Acceleration due to Gravity (vertical motion)

Free fall: when only gravity acts on an object (acceleration is due to gravity)

Objects in free fall have a constant acceleration. This acceleration is due to gravity.

Acceleration due to gravity is a vector.

Its **magnitude** is 9.8 m/s^2.

Its **directions** is DOWN (toward the centre of the Earth).

Note: We can choose + to be up or down. We have to make sure we are always consistent with our signs (i.e. all values going "up" have the same sign, all values going "down" have the same sign).

Different possible (useful) situations

An object is dropped.

$v_i = 0$
a = down
v_f = down
Δd = down

choose ↓ +

An object is thrown down.

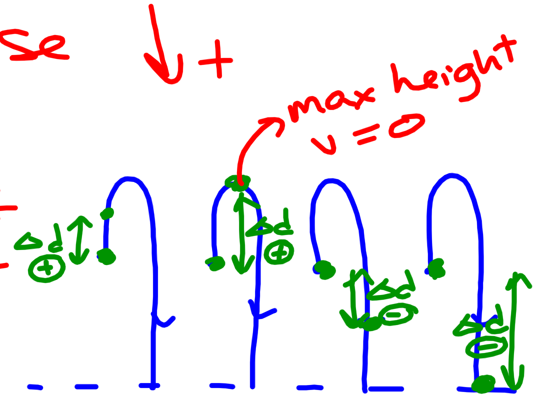
v_i = down
a = down
v_f = down
Δd = down

choose ↓ +

An object is thrown upward.

v_i = up *
a = down *
v_f = up or down
Δd = up or down

choose ↑ + ↓ −



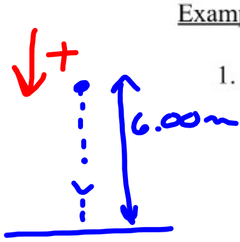
An object is thrown up and caught at the same height.

$\Delta d = 0$
a = down
v_i = up
v_f = down

*choose ↑ + ↓ −

v_i and v_f have same magnitude, but opposite directions.

Examples:



1. You drop a rock from a window 6.00 m above the ground. How long does it take for the rock to hit the ground?

$v_i = 0$

$v_i = 0$
$a = 9.8\,m/s^2$
$\Delta d = 6.00\,m$
$\Delta t = ?$

$$\Delta d = \cancel{v_i \Delta t}^{\,0} + \frac{1}{2} a (\Delta t)^2$$

$$\Delta d = \frac{1}{2} a (\Delta t)^2$$

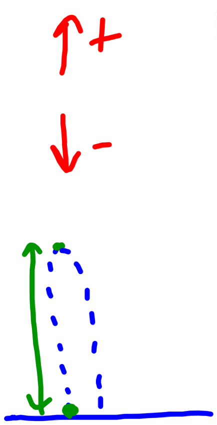
$$(\Delta t)^2 = \frac{\Delta d}{\frac{1}{2} a}$$

$$\Delta t = \sqrt{\frac{6.00\,m}{\frac{1}{2}(9.8\,\frac{m}{s^2})}}$$

$$= 1.11\,s$$

units

$$\sqrt{\frac{m}{\frac{m}{s^2}}} = \sqrt{\cancel{m} \times \frac{s^2}{\cancel{m}}} = s$$



2. A rock is launched directly upward from the ground at a speed of 12.0 m/s. What is the maximum height reached by the rock?

$v_i = +12.0\,m/s$
$a = -9.8\,m/s^2$
$\Delta d = ?$
$v_f = 0$

$$\cancel{v_f^2}^{\,0} = v_i^2 + 2a\Delta d$$

$$-v_i^2 = 2a\Delta d$$

$$\Delta d = \frac{-v_i^2}{2a}$$

$$= \frac{+(12.0\,m/s)^2}{2(9.8\,m/s^2)}$$

$$\Delta d = 7.35\,m$$

$$\frac{\frac{m^2}{s^2}}{\frac{m}{s^2}} = \frac{m^2}{\cancel{s^2}} \times \frac{\cancel{s^2}}{\cancel{m}}$$

$\uparrow +$

$\downarrow -$

3. You toss a ball up in the air, giving it an initial speed of 8.00 m/s. You catch the ball at the same height as you tossed it. How long does the ball stay in the air?

Method #1 ($\Delta d = 0$)

$v_i = +8.00\ m/s$

$\cancel{\Delta d = 0}$

$a = -9.8\ m/s^2$

$\Delta t = ?$

$\cancel{\Delta d}^{\,0} = v_i \Delta t + \frac{1}{2}a(\Delta t)^2$

$0 = v_i \Delta t + \frac{1}{2}a(\Delta t)^2$

$0 = \Delta t(v_i + \frac{1}{2}a\Delta t)$

$\Delta t = 0 \leftarrow$ start!

$v_i + \frac{1}{2}a\Delta t = 0$

$\frac{1}{2}a\Delta t = -v_i$

$\Delta t = \frac{-v_i}{\frac{1}{2}a} = \frac{-8.00\ m/s}{\frac{1}{2}(-9.8\ m/s^2)} = 1.63\ s$

Method #2 ($\cancel{v_f = -v_i}$)

$\cancel{v_i} = +8.00\ m/s$

$\cancel{v_f} = -8.00\ m/s$

$a = -9.8\ m/s^2$

$\Delta t = ?$

$v_f = v_i + a\Delta t$

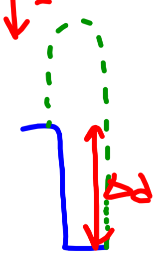
$\Delta t = \frac{v_f - v_i}{a}$

$= \frac{-8.00\ m/s - 8.00\ m/s}{-9.8\ m/s^2}$

$= 1.63\ s$

4. Standing on the edge of the roof of a building 7.00 m tall, you toss a ball straight up at a speed of 5.00 m/s. How fast will the ball be moving when it hits the ground?

$\uparrow +$ $\downarrow -$



$v_i = 5.00\ m/s$

$\Delta d = -7.00\ m$

$a = -9.8\ m/s^2$

$v_f = ?$

$v_f^2 = v_i^2 + 2a\Delta d$

$v_f^2 = (5.00\ \frac{m}{s})^2 + 2(-9.8\ \frac{m}{s^2})(-7.00\ m)$

$= 25.0\ \frac{m^2}{s^2} + 137.2\ \frac{m^2}{s^2}$

$\sqrt{v_f^2} = \sqrt{162.2\ \frac{m^2}{s^2}}$

$= \pm 12.7\ \frac{m}{s}$

$\boxed{v_f = -12.7\ m/s}$

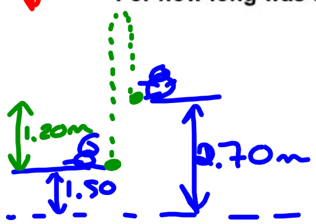
Physics 504

Choosing between positive times (in freefall problems)

↑ +

↓ −

A monkey at the zoo is standing on a branch 1.50 m above the ground when he throws a banana straight up in the air at a speed of 6.00 m/s. Another monkey, sitting on a branch 2.70 m above the ground, sees the banana on its way up, then catches it on its way down. For how long was the banana in the air, from when it was thrown to when it was caught?



$v_i = 6.00\,m/s$

$a = -9.8\,m/s^2$

$\Delta d = 1.20\,m$

$\Delta t = ?$

Method #1

$$\Delta d = v_i \Delta t + \frac{1}{2} a (\Delta t)^2$$

quadratic

$$1.20 = (6.00)\Delta t + \frac{1}{2}(-9.8)(\Delta t)^2$$

$$0 = \underbrace{-4.9}_{a}(\Delta t)^2 + \underbrace{(6.00)}_{b}\Delta t \underbrace{- 1.20}_{c}$$

$$\Delta t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta t = \frac{-6.00 \pm \sqrt{(6.00)^2 - 4(-4.9)(-1.20)}}{2(-4.9)}$$

$$= \frac{-6.00 \pm \sqrt{12.48}}{-9.8}$$

$$= \frac{-6.00 \pm 3.53}{-9.8}$$

+ : $$\Delta t = \frac{-6.00 + 3.53}{-9.8}$$

$\Delta t = \cancel{0.25\,s}$

way up
(smaller time)

− : $$\Delta t = \frac{-6.00 - 3.53}{-9.8}$$

$\boxed{\Delta t = 0.97\,s}$

way down
(larger time

Method #2 (find v_f)

$$v_f^2 = v_i^2 + 2a\Delta d$$

$$v_f^2 = (6.00\,\tfrac{m}{s})^2 + 2(-9.8\,\tfrac{m}{s^2})(1.20$$

$$v_f^2 = 36.0\,\tfrac{m^2}{s^2} - 23.52\,\tfrac{m^2}{s^2}$$

$$v_f^2 = 12.48\,\tfrac{m^2}{s^2}$$

$$v_f = \pm\sqrt{12.48\,\tfrac{m^2}{s^2}}$$

$$v_f = -3.53\,\tfrac{m}{s}$$

v_f + way up

v_f − way down

Catch-up / Meeting Problems (2 bodies in motion)

There is no single way of solving these problems. However, here are a few starting points:

- Use *x* and *y* for your unknowns (for both bodies in motion).
- Ask yourself what is the same for both bodies.

Examples: Car related

1. A car moving at 12 m/s passes a truck that is stationary. At that moment, the truck begins to accelerate at 2.0 m/s^2. How long does it take for the truck to catch up to the car?

Car

$v = 12\ m/s$

$\Delta t = x$ ← same → $\Delta t = x$

$\Delta d = y$ ← same → $\Delta d = y$

Truck

$a = 2.0\ m/s^2$

$v_i = 0$

$\Delta t = x$

$\Delta d = y$

$\Delta d = v\Delta t$

$\boxed{y = (12\ \tfrac{m}{s})x}$

system!

$\Delta d = \cancel{v_i \Delta t}^{\,0} + \frac{1}{2}a(\Delta t)^2$

$y = \frac{1}{2}(2.0\ \tfrac{m}{s^2})(x)^2$

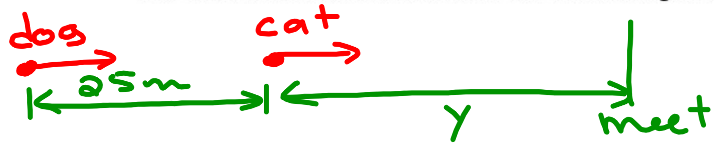
$\boxed{y = (1\ \tfrac{m}{s^2})x^2}$

$\frac{\cancel{m}}{\cancel{s}} \times \frac{s^{\cancel{2}}}{\cancel{m}}$

$(12\ \tfrac{m}{s})x = (1\ \tfrac{m}{s^2})x^2$

$\boxed{12\ s = x}$

2. A dog is running at a constant 8.0 m/s and a cat is running at a constant 5.0 m/s. The cat has a 25 m head start. How far will the dog run before it catches the cat?



Cat

$\Delta d = y$

$v = 5.0\,m/s$

$\Delta t = x$

$\Delta d = v\Delta t$

$\boxed{y = (5.0\,\tfrac{m}{s})x}$

system

Dog

$\Delta d = \cancel{y + 25\,m}$

$v = 8.0\,m/s$

$\Delta t = x$

$\Delta d = v\Delta t$

$\boxed{y + 25\,m = (8.0\,\tfrac{m}{s})x}$

$$(5.0\,\tfrac{m}{s})x + 25\,m = (8.0\,\tfrac{m}{s})x$$

$$25\,m = (8.0\,\tfrac{m}{s})x - (5.0\,\tfrac{m}{s})x$$

$$25\,m = (3.0\,\tfrac{m}{s})x$$

$$x = \frac{25\,m}{3.0\,\frac{m}{s}} \qquad \cancel{m} \times \frac{s}{\cancel{m}}$$

$$x = 8.3\dot{3}\,s$$

Δd of dog

$$\Delta d = v\Delta t$$
$$= (8.0\,\tfrac{m}{s})(8.3\dot{3}\,s)$$
$$= 67\,m$$

$y = ?$

$$y = (5.0\,\tfrac{m}{s})x$$
$$= (5.0\,\tfrac{m}{s})(8.3\dot{3}\,s)$$
$$= 41.7\,m$$

$$\Delta d \text{ dog} = y + 25\,m$$
$$= 41.7\,m + 25\,m$$
$$= 67\,m$$